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New time-type and space-type non-standard quantum algebras and discrete symmetries

Francisco J Herranz

Departamento de Física, Escuela Politécnica Superior, Universidad de Burgos, E-09006 Burgos, Spain

Received 6 September 2000

Abstract. Starting from the classical r matrix of the non-standard (or Jordanian) quantum deformation of the $sl(2, \mathbb{R})$ algebra, new triangular quantum deformations for the real Lie algebras $so(2, 2)$, $so(3, 1)$ and $iso(2, 1)$ are simultaneously constructed by using a graded contraction scheme; these are realized as deformations of conformal algebras of $(1+1)$ -dimensional spacetimes. Time- and space-type quantum algebras are considered according to the generator that remains primitive after deformation: either the time or the space translation, respectively. Furthermore, by introducing differential-difference conformal realizations, these families of quantum algebras are shown to be the symmetry algebras of either a time or a space discretization of $(1+1)$ -dimensional (wave and Laplace) equations on uniform lattices; the relationship with the known Lie symmetry approach to these discrete equations is established by means of twist maps.

1. Introduction

The non-standard (or Jordanian) quantum deformation of $sl(2, \mathbb{R}) \simeq so(2, 1)$ [1], $U_z(sl(2, \mathbb{R}))$, has been the starting point in the construction of non-standard (or triangular) quantum algebras in higher dimensions. In particular, by taking two copies of $U_z(sl(2, \mathbb{R}))$ and applying the same procedure as in the standard (Drinfel'd–Jimbo) case [2], a quantum $so(2, 2)$ algebra has been obtained in [3], while the corresponding deformation for $so(3, 2)$ has been found in [4]. These quantum algebras have been realized as deformations of conformal algebras for the Minkowskian spacetime. Furthermore, by following either a contraction approach [3] or a deformation embedding method [5], non-standard quantum deformations for other Lie algebras have been deduced; amongst them it is remarkable it has the appearance of a non-standard quantum Poincaré algebra, which can be considered as a quantum conformal algebra for the Carroll spacetime, or alternatively and more interestingly, as a null-plane quantum Poincaré algebra [5, 6]. All these results are summarized in the following diagram where the vertical arrows indicate the contractions leading to quantum Poincaré algebras:

$$\begin{array}{ccccc}
 U_z(sl(2, \mathbb{R})) & \longrightarrow & U_z(sl(2, \mathbb{R})) \oplus U_{-z}(sl(2, \mathbb{R})) \simeq U_z(so(2, 2)) & \longrightarrow & U_z(so(3, 2)) \\
 \downarrow \varepsilon \rightarrow 0 & & \downarrow \varepsilon \rightarrow 0 & & \downarrow \varepsilon \rightarrow 0 \\
 U_z(iso(1, 1)) & \longrightarrow & \text{Null-plane Poincaré algebra} & \longrightarrow & U_z(iso(3, 1))
 \end{array}$$

The first aim of this paper is to provide, starting again from $U_z(sl(2, \mathbb{R}))$, a new way for the construction of non-standard quantum algebras, obtaining a new non-standard quantum $so(2, 2)$ algebra which could be the cornerstone of further constructions in higher dimensions. The essential idea is to require that $U_z(sl(2, \mathbb{R}))$ remains as a Hopf subalgebra, or to be more

precise, to keep its underlying Jordanian classical r matrix, $r = zJ_3 \wedge J_+$, as the element generating the whole deformation for $so(2, 2)$. Hence this approach can be seen as a kind of *complete* deformation embedding method leading to $U_z(sl(2, \mathbb{R})) \subset U_z(so(2, 2))$, so that this seems to be a more feasible and applicable quantum deformation procedure than the involved one used in [4] for $so(3, 2)$ when the extension to higher dimensions is attacked.

Two choices for such a Jordanian classical r matrix associated with $so(2, 2)$ naturally appear: one gives rise to a *time-type* quantum deformation characterized by a primitive generator of time translations; meanwhile the other leads to a *space-type* deformation determined by a primitive generator of space translations; the Drinfel'd–Jimbo counterpart of these types of deformations can be found in [7]. Furthermore, by using graded contractions this task is carried out for the real Lie algebras $so(2, 2)$, $so(3, 1)$ and $iso(2, 1)$, simultaneously; the quantum algebras so obtained are realized as deformations of conformal algebras of $(1+1)$ D spacetimes.

The second aim of this paper is to analyse the discrete symmetries provided by both families of quantum algebras as differential-difference conformal operators of either a time or a space discretization of some $(1+1)$ D differential equations (the wave and Laplace equations) on a uniform lattice, and next to relate these results with the Lie symmetry analysis presented in [8]. This objective is achieved by following a similar procedure to the one used in [9] with respect to non-standard quantum Schrödinger algebras and their associated discrete symmetries.

The structure of the paper is as follows. We summarize in the next section the basic aspects of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded contractions of $so(2, 2)$ in a conformal basis as well as their role in symmetry algebras of $(1+1)$ D differential equations. The construction of the time-type quantum deformation, together with its universal R matrix, is developed in section 3. These quantum algebras are shown to be the symmetry algebras of a time discretization of the wave and Laplace equations on a uniform lattice in section 4; the relationship with the Lie symmetry approach studied in [8] is established by means of a twist map. A parallel procedure with the space-type quantum deformation is carried out in section 5. Finally, an algebraic equivalence or *duality* between both types of quantum algebras is introduced in the last section where we also comment on their possible generalization to higher dimensions and the way of obtaining new null-plane quantum Poincaré algebras.

2. Graded contractions of $so(2, 2)$ and continuous symmetries

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded contractions of the real Lie algebra $so(2, 2)$ have been analysed in [3], where a distinguished set of solutions has been explicitly considered and expressed in terms of three contraction parameters (μ_1, μ_2, μ_3) . Here we shall restrict ourselves to dealing with the most relevant contracted Lie algebras setting $(\mu_1, \mu_2, \mu_3) = (\mu, +1, \nu)$, so that all of them are collectively denoted $so_{\mu, \nu}(2, 2)$. Recall that each contraction parameter can take either a positive, zero or negative value and whenever they are different from zero they can be scaled to ± 1 .

At this dimension a generic Lie algebra in the family $so_{\mu, \nu}(2, 2)$ can be interpreted in two different frameworks: either as the algebra of isometries of a $(2+1)$ D spacetime (or a 3D space), or as the algebra of conformal transformations of a $(1+1)$ D spacetime (or a 2D space). In this paper we will adopt the latter interpretation; hence let us consider the generators of time translations H , space translations P , boosts K , dilations D and special conformal transformations C_1, C_2 . In this basis, the Lie brackets of the set of graded contractions

$so_{\mu,\nu}(2, 2)$ are

$$\begin{aligned}
 [K, H] &= \nu P & [K, P] &= \mu H & [H, P] &= 0 \\
 [D, H] &= H & [D, C_1] &= -C_1 & [H, C_1] &= -2\nu D \\
 [D, P] &= P & [D, C_2] &= -C_2 & [P, C_2] &= 2\mu D \\
 [K, C_1] &= \nu C_2 & [K, C_2] &= \mu C_1 & [K, D] &= 0 \\
 [H, C_2] &= 2K & [P, C_1] &= -2K & [C_1, C_2] &= 0.
 \end{aligned} \tag{2.1}$$

The two Casimirs of $so_{\mu,\nu}(2, 2)$ turn out to be

$$\begin{aligned}
 W_1 &= K^2 + \mu\nu D^2 - \frac{1}{2}\mu(HC_1 + C_1H) + \frac{1}{2}\nu(PC_2 + C_2P) \\
 W_2 &= KD + \frac{1}{2}(HC_2 - C_1P).
 \end{aligned} \tag{2.2}$$

In what follows we identify each specific real Lie algebra appearing within the family $so_{\mu,\nu}(2, 2)$ (see table 1 below) and comment on its physical (or geometrical) role according to the (signs or zero) values of the pair (μ, ν) [3]:

- $so(2, 2)$ when $(\mu, \nu) \in \{(+, +), (-, -)\}$. This is the conformal algebra of the (1 + 1)D Minkowskian spacetime; alternatively, it can be seen as the kinematical algebra of the (2 + 1)D anti-de Sitter spacetime.
- $so(3, 1)$ when $(\mu, \nu) \in \{(+, -), (-, +)\}$. This is the conformal algebra of the 2D Euclidean space so that, under this interpretation, H should be considered as another generator of space translations. This algebra can also be realized as the kinematical algebra of the (2 + 1)D de Sitter spacetime.
- $iso(2, 1)$ when $(\mu, \nu) \in \{(+, 0), (0, +), (-, 0), (0, -)\}$. In these four cases, this is the kinematical algebra of the (2 + 1)D Minkowskian spacetime, that is, the (2 + 1)D Poincaré algebra. However, this corresponds to the conformal algebra of the (1 + 1)D Galilean spacetime whenever $(\mu, \nu) = (0, \pm)$, but to the conformal algebra of the (1 + 1)D Carroll spacetime whenever $(\mu, \nu) = (\pm, 0)$.
- $iiso(1, 1)$ when $(\mu, \nu) = (0, 0)$. This is the most contracted algebra in the family $so_{\mu,\nu}(2, 2)$ and has no known conformal interpretation, although it is the algebra of isometries of certain 3D spaces. Note that, in this case, K is a central generator.

The aforementioned conformal role of the algebras $so_{\mu,\nu}(2, 2)$ (with the exception of $iiso(1, 1)$) can be appreciated more clearly by taking into account that: (i) the Lie brackets of the subalgebra spanned by $\{K, H, P\}$ generate the algebra of isometries of the corresponding (1 + 1)D spacetime (or 2D Euclidean space). (ii) When the dilation generator is added, we find the so-called Weyl subalgebra $\{K, H, P, D\}$ which is the similitude algebra of the (1 + 1)D spacetime. (iii) If conformal transformations are also considered, then we obtain the complete conformal Lie group $SO_{\mu,\nu}(2, 2)$; its quotient with the subgroup generated by $\{K, C_1, C_2, D\}$ is identified with the (1 + 1)D conformal spacetime.

The relationship between $so_{\mu,\nu}(2, 2)$ and (1 + 1)D differential equations can be established by considering the usual (conformal) vector field representation in terms of the space and time coordinates (x, t) :

$$\begin{aligned}
 H &= \partial_t & P &= \partial_x & K &= -\nu t \partial_x - \mu x \partial_t & D &= -x \partial_x - t \partial_t \\
 C_1 &= (\mu x^2 + \nu t^2) \partial_t + 2\nu x t \partial_x & C_2 &= -(\mu x^2 + \nu t^2) \partial_x - 2\mu x t \partial_t
 \end{aligned} \tag{2.3}$$

where we exclude the degenerate case $iiso(1, 1)$ with $\mu = \nu = 0$. This is a zero-value realization of the two Casimirs (2.2). The action of the Casimir of the Lie subalgebra $\{K, H, P\}$

$$E = \nu P^2 - \mu H^2 \tag{2.4}$$

on a function $\Phi(x, t)$ through the representation (2.3) (choosing for E the zero eigenvalue) leads to the following (1 + 1)D differential equation:

$$E\Phi(x, t) = 0 \implies \left(v \frac{\partial^2}{\partial x^2} - \mu \frac{\partial^2}{\partial t^2} \right) \Phi(x, t) = 0. \quad (2.5)$$

We shall say that an operator \mathcal{O} is a symmetry of the equation $E\Phi(x, t) = 0$ if \mathcal{O} transforms solutions into solutions, that is, $E\mathcal{O} = \Lambda E$ where Λ is another operator. Hence, $so_{\mu, \nu}(2, 2)$ is the symmetry algebra of the equation (2.5), since E given by (2.4) commutes with $\{K, H, P\}$ and in the realization (2.3) the remaining generators are also symmetry operators of (2.5) verifying

$$[E, D] = -2E \quad [E, C_1] = 4\nu t E \quad [E, C_2] = -4\mu x E. \quad (2.6)$$

From this perspective, we find that the equation (2.5) reproduces the (1 + 1)D wave equation when the contraction parameters (μ, ν) are either $(+, +)$ or $(-, -)$, which in turn means that $so(2, 2)$ is its associated algebra of symmetry operators. Likewise, $so(3, 1)$ corresponding to $(+, -)$ or $(-, +)$ arises as the symmetry algebra of the 2D Laplace equation (in this case t should be seen as another space coordinate). Finally, the contraction with either $\mu = 0$ or $\nu = 0$ leads to the 1D Laplace equation with $iso(2, 1)$ as its symmetry Lie algebra.

3. Time-type quantum algebras

Let us consider the subalgebra of $so_{\mu, \nu}(2, 2)$ spanned by $\{D, H\}$ with Lie bracket $[D, H] = H$, and the non-standard or Jordanian classical r matrix given by [10, 11]

$$r = -\tau D \wedge H \quad (3.1)$$

which is a solution of the classical Yang–Baxter equation and τ is the deformation parameter. As is well known, the deformed commutator and coproduct for this subalgebra can be written as

$$[D, H] = \frac{1 - e^{-\tau H}}{\tau} \quad \Delta(H) = 1 \otimes H + H \otimes 1 \quad \Delta(D) = 1 \otimes D + D \otimes e^{-\tau H}. \quad (3.2)$$

We recall that this structure is a Hopf subalgebra of non-standard quantum deformations of $sl(2, \mathbb{R})$ [1, 12–15], $iso(1, 1)$, $gl(2)$, h_4 and Schrödinger algebras [9]; this was also introduced in [16, 17] in relation to an approach to physics at the Planck scale.

If we now impose the classical r matrix (3.1) to be the generating object of a quantum deformation for the whole family $so_{\mu, \nu}(2, 2)$, then the cocommutator δ of a generator X that defines the associated Lie bialgebra is obtained as $\delta(X) = [1 \otimes X + X \otimes 1, r]$, namely

$$\begin{aligned} \delta(H) &= 0 & \delta(D) &= -\tau D \wedge H \\ \delta(P) &= \tau P \wedge H & \delta(K) &= -\tau \nu D \wedge P \\ \delta(C_1) &= -\tau C_1 \wedge H & \delta(C_2) &= -\tau C_2 \wedge H + 2\tau D \wedge K. \end{aligned} \quad (3.3)$$

The coproduct Δ for the quantum algebras denoted $U_\tau(so_{\mu, \nu}(2, 2))$ is obtained by solving the coassociativity condition $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$, by requiring that (3.2) remains as a Hopf subalgebra of $U_\tau(so_{\mu, \nu}(2, 2))$, and by taking into account that δ is related to the first order of Δ on τ , $\Delta_{(1)}$, by $\delta = \Delta_{(1)} - \sigma \circ \Delta_{(1)}$ where $\sigma(X \otimes Y) = Y \otimes X$. The resulting coproduct turns out to be

$$\begin{aligned} \Delta(H) &= 1 \otimes H + H \otimes 1 & \Delta(D) &= 1 \otimes D + D \otimes e^{-\tau H} \\ \Delta(P) &= 1 \otimes P + P \otimes e^{\tau H} & \Delta(C_1) &= 1 \otimes C_1 + C_1 \otimes e^{-\tau H} \\ \Delta(K) &= 1 \otimes K + K \otimes 1 - \tau \nu D \otimes e^{-\tau H} P \\ \Delta(C_2) &= 1 \otimes C_2 + C_2 \otimes e^{-\tau H} + 2\tau D \otimes e^{-\tau H} K - \tau^2 \nu (D^2 + D) \otimes e^{-2\tau H} P. \end{aligned} \quad (3.4)$$

Thereafter, the deformed commutation rules are deduced by imposing Δ to be an algebra homomorphism, that is, $\Delta([X, Y]) = [\Delta(X), \Delta(Y)]$; they are

$$\begin{aligned}
 [K, H] &= \nu e^{-\tau H} P & [K, P] &= \mu \frac{e^{\tau H} - 1}{\tau} \\
 [H, P] &= 0 & [K, D] &= 0 \\
 [D, H] &= \frac{1 - e^{-\tau H}}{\tau} & [D, C_1] &= -C_1 + \tau \nu D^2 \\
 [H, C_1] &= -2\nu D & [D, P] &= P \\
 [D, C_2] &= -C_2 & [P, C_2] &= 2\mu D \\
 [K, C_1] &= \nu C_2 & [K, C_2] &= \mu C_1 - \tau \mu \nu D^2 \\
 [H, C_2] &= e^{-\tau H} K + K e^{-\tau H} & [P, C_1] &= -2K - \tau \nu (DP + PD) \\
 [C_1, C_2] &= -\tau \nu (DC_2 + C_2 D).
 \end{aligned} \tag{3.5}$$

The deformed Casimirs of $U_\tau(so_{\mu,\nu}(2, 2))$ are given by

$$\begin{aligned}
 W_{1,\tau} &= K^2 + \mu \nu D^2 - \frac{1}{2} \mu \left(\frac{e^{\tau H} - 1}{\tau} C_1 + C_1 \frac{e^{\tau H} - 1}{\tau} \right) + \frac{1}{2} \nu (PC_2 + C_2 P) \\
 &\quad + \frac{1}{2} \mu \nu (e^{\tau H} D^2 + D^2 e^{\tau H}) - \mu \nu D^2 \\
 W_{2,\tau} &= KD + \frac{1}{2} \left(\frac{e^{\tau H} - 1}{\tau} C_2 - C_1 P \right) + \frac{1}{2} \tau \nu D^2 P.
 \end{aligned} \tag{3.6}$$

Mathematical and physical properties of the quantum algebras $U_\tau(so_{\mu,\nu}(2, 2))$ are characterized by their *primitive generator* H (that with a vanishing cocommutator). More explicitly, since the product τH has to be dimensionless in order to have a homogeneous coproduct, the deformation parameter τ has, in principle, the dimension of a *time* (notice that for $U_\tau(so(3, 1))$, within the conformal interpretation, τ would be a length). This is similar to what happens with the well known κ -Poincaré algebra [18–20], realized as a kinematical algebra of the Minkowskian spacetime, and where the time translation generator is also primitive; the deformation parameters κ and τ would be related by $\kappa = 1/\tau$. Furthermore, as we shall show in the next section, these time-type quantum algebras directly lead to a time discretization of the symmetries (2.3) and equation (2.5) on a uniform lattice.

On the other hand, at the level of Hopf subalgebras of $U_\tau(so_{\mu,\nu}(2, 2))$ two remarkable structures arise:

- The generators $\{D, H, C_1\}$ close a Hopf subalgebra isomorphic either to the Jordanian quantum $sl(2, \mathbb{R}) \simeq so(2, 1)$ algebra if $\nu \neq 0$, or to a non-standard quantum Poincaré algebra $U_\tau(iso(1, 1))$ under the contraction $\nu = 0$.
- The generators $\{K, H, P, D\}$ span a Hopf subalgebra which is the similitude algebra of a (1 + 1)D spacetime. Therefore, as a byproduct of our construction, we obtain for each member in the family $U_\tau(so_{\mu,\nu}(2, 2))$ a new quantum deformation of the Weyl subalgebra of the corresponding conformal algebra. Notice that $\{K, H, P\}$ only close a Hopf subalgebra whenever $\nu = 0$.

For the sake of clarity the specific Hopf subalgebras spanned by $\{D, H, C_1\}$ and $\{K, H, P, D\}$ for each quantum algebra in the family $U_\tau(so_{\mu,\nu}(2, 2))$ are displayed in table 1 according to the values of the pair (μ, ν) . The horizontal arrows indicate the contraction $\mu = 0$ and the vertical ones the contraction $\nu = 0$. The symbols \mathcal{WM} , \mathcal{WE} , \mathcal{WG} and \mathcal{WC} mean, in this order, the Weyl subalgebra of the Minkowskian, Euclidean, Galilean and Carroll planes, thus reminding one of the corresponding conformal spaces, while \mathcal{WA} means the Abelian

Table 1. Hopf subalgebras $\{D, H, C_1\}$ and $\{K, H, P, D\}$, and associated (difference and/or differential) equations of the time-type quantum algebras $U_\tau(so_{\mu,\nu}(2, 2))$.

$(+, +) \quad U_\tau(so(2, 2))$ $U_\tau(sl(2, \mathbb{R})) \quad U_\tau(\mathcal{WM})$ $(\partial_x^2 - \Delta_t^2)\Phi = 0$	\longrightarrow	$(0, +) \quad U_\tau(iso(2, 1))$ $U_\tau(sl(2, \mathbb{R})) \quad U_\tau(\mathcal{WG})$ $\partial_x^2\Phi = 0$	\longleftarrow	$(-, +) \quad U_\tau(so(3, 1))$ $U_\tau(sl(2, \mathbb{R})) \quad U_\tau(\mathcal{WE})$ $(\partial_x^2 + \Delta_t^2)\Phi = 0$
\downarrow		\downarrow		\downarrow
$(+, 0) \quad U_\tau(iso(2, 1))$ $U_\tau(iso(1, 1)) \quad U_\tau(\mathcal{WC})$ $\Delta_t^2\Phi = 0$	\longrightarrow	$(0, 0) \quad U_\tau(i'iso(1, 1))$ $U_\tau(iso(1, 1)) \quad U_\tau(\mathcal{VA})$ Degenerate equation	\longleftarrow	$(-, 0) \quad U_\tau(iso(2, 1))$ $U_\tau(iso(1, 1)) \quad U_\tau(\mathcal{WC})$ $\Delta_t^2\Phi = 0$
\uparrow		\uparrow		\uparrow
$(+, -) \quad U_\tau(so(3, 1))$ $U_\tau(sl(2, \mathbb{R})) \quad U_\tau(\mathcal{VE})$ $(\partial_x^2 + \Delta_t^2)\Phi = 0$	\longrightarrow	$(0, -) \quad U_\tau(iso(2, 1))$ $U_\tau(sl(2, \mathbb{R})) \quad U_\tau(\mathcal{VG})$ $\partial_x^2\Phi = 0$	\longleftarrow	$(-, -) \quad U_\tau(so(2, 2))$ $U_\tau(sl(2, \mathbb{R})) \quad U_\tau(\mathcal{VM})$ $(\partial_x^2 - \Delta_t^2)\Phi = 0$

algebra enlarged with a dilation generator. In this context, we remark that other non-standard quantum deformations of these Weyl algebras have been carried out in [21], the underlying classical r matrix of which is, in our notation, $r = \omega(K \wedge H + D \wedge P)$; their generalization to higher dimensions can be found in [4]. We also recall that other non-standard classical r matrices for $so(3, 2)$ and $so(4, 2)$ (expressed as conformal algebras) can be found in [22].

3.1. Universal quantum R matrix

Different constructions of the universal quantum R matrix associated with the non-standard quantum deformation of the Borel algebra (of the type $[D, H] = H$) have appeared in the literature [14, 15, 23, 24], mainly in relation to the Jordanian quantum $sl(2, \mathbb{R})$ algebra. If we consider the quantum Borel algebra written in the form of (3.2), then the universal R matrix turns out to be [24]

$$\mathcal{R} = \exp\{\tau H \otimes D\} \exp\{-\tau D \otimes H\} \tag{3.7}$$

which is a solution of the quantum Yang–Baxter equation and also fulfils

$$\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \sigma \circ \Delta(X) \quad \text{for } X \in \{H, D\}. \tag{3.8}$$

As could be expected, the element (3.7) is a triangular universal R matrix for the *whole* family $U_\tau(so_{\mu,\nu}(2, 2))$ since it contains (3.2) as a Hopf subalgebra and the four remaining generators also verify (3.8)

$$\begin{aligned} \exp\{-\tau D \otimes H\}\Delta(C_1)\exp\{\tau D \otimes H\} &= 1 \otimes C_1 + C_1 \otimes 1 + 2\tau\nu D \otimes D \equiv f \\ \exp\{\tau H \otimes D\}f\exp\{-\tau H \otimes D\} &= \sigma \circ \Delta(C_1) \end{aligned} \tag{3.9}$$

$$\begin{aligned} \exp\{-\tau D \otimes H\}\Delta(X)\exp\{\tau D \otimes H\} &= 1 \otimes X + X \otimes 1 \equiv \Delta_0(X) \\ \exp\{\tau H \otimes D\}\Delta_0(X)\exp\{-\tau H \otimes D\} &= \sigma \circ \Delta(X) \quad \text{for } X \in \{P, K, C_2\}. \end{aligned} \tag{3.10}$$

The lower-dimensional matrix representation of $U_\tau(so_{\mu,\nu}(2, 2))$ is given by the following 4×4 real matrices:

$$\begin{aligned}
 H &= \begin{pmatrix} \frac{1}{2}\tau\nu & -\frac{1}{2}\tau\nu & -\nu & 0 \\ \frac{1}{2}\tau\nu & -\frac{1}{2}\tau\nu & -\nu & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & P &= \begin{pmatrix} 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \\
 K &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & \nu & 0 \end{pmatrix} & D &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 C_1 &= \begin{pmatrix} \tau\nu & 0 & -\nu & 0 \\ 0 & \tau\nu & \nu & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & C_2 &= \begin{pmatrix} 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & -\mu \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.
 \end{aligned} \tag{3.11}$$

We exclude the most contracted quantum algebra with $\mu = \nu = 0$, since in this case the matrix of K is degenerate (and this is a central generator). This representation allows us to deduce a 16×16 matrix expression for \mathcal{R} . Let us denote by $\mathbf{1}$ and $\mathbf{0}$ the 4×4 unit and zero matrices; under the representation (3.11) we find that $H^3 = \mathbf{0}$ so that the quantum \mathcal{R} matrix (3.7) reduces to

$$\mathcal{R} = (\mathbf{1} \otimes \mathbf{1} + \tau H \otimes D + \frac{1}{2}\tau^2 H^2 \otimes D^2)(\mathbf{1} \otimes \mathbf{1} - \tau D \otimes H + \frac{1}{2}\tau^2 D^2 \otimes H^2) \tag{3.12}$$

which can finally be written in block-matrix form as $\mathcal{R} =$

$$\left(\begin{array}{cccc|cccc|cccc|c}
 1 - \tau^2\nu & \tau^2\nu & 0 & 0 & 0 & 0 & \tau\nu & 0 & 0 & -\tau\nu & 0 & 0 & \mathbf{0} \\
 0 & 1 & 0 & 0 & -\tau^2\nu & \tau^2\nu & \tau\nu & 0 & -\tau\nu & 0 & 0 & 0 & \mathbf{0} \\
 0 & 0 & 1 & 0 & -\tau & \tau & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
 \hline
 -\tau^2\nu & \tau^2\nu & \tau\nu & 0 & 1 & 0 & 0 & 0 & 0 & -\tau\nu & 0 & 0 & \mathbf{0} \\
 0 & 0 & \tau\nu & 0 & -\tau^2\nu & 1 + \tau^2\nu & 0 & 0 & -\tau\nu & 0 & 0 & 0 & \mathbf{0} \\
 -\tau & \tau & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \mathbf{0} \\
 \hline
 0 & \tau & -\tau^2\nu & 0 & 0 & -\tau & \tau^2\nu & 0 & 1 & 0 & 0 & 0 & \mathbf{0} \\
 \tau & 0 & -\tau^2\nu & 0 & -\tau & 0 & \tau^2\nu & 0 & 0 & 1 & 0 & 0 & \mathbf{0} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{0} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{0} \\
 \hline
 \mathbf{0} & & & & \mathbf{0} & & & & \mathbf{0} & & & & \mathbf{1}
 \end{array} \right). \tag{3.13}$$

So far these last results could be further exploited in different directions. On the one hand, the expression (3.7) should allow one to obtain a triangular R -matrix solution of the coloured Yang–Baxter equation (that is, with spectral parameters); indeed, this is formally rather similar to the universal R matrix of the Jordanian $gl(2)$ algebra used in [25] in order to deduce its coloured realization. On the other hand, the matrices (3.11) and (3.13) could be applied in the computation of the differential calculus on the quantum conformal spaces associated with $U_\tau(so_{\mu,\nu}(2, 2))$. In this respect see, for instance, [26] where the construction of the quantum anti-de Sitter space from the quantum algebra $SO_q(3, 2)$ (of Drinfel’d–Jimbo type) has been carried out. The explicit presence of the contraction parameters would enable a simultaneous study of these problems for $so(2, 2)$, $so(3, 1)$ and $iso(2, 1)$ with a built-in scheme of contractions.

4. Discrete time symmetries

Non-standard quantum Schrödinger algebras have been recently shown to be the Hopf algebras of symmetries of a time (or a space) discretization of the heat-Schrödinger equation on a uniform lattice [9]; in that construction, the deformation parameter plays the role of the time (or space) lattice constant. Furthermore, by making use of twist maps those discrete Schrödinger symmetries obtained from quantum algebras have been connected with the discretization (in a single variable) of the heat-Schrödinger equation deduced in [27] by following the usual Lie symmetry theory. In this context, the remarkable point is that the same classical procedure has also been applied in [8] to the study of the symmetries of a discretization of the (1 + 1)D wave equation in both coordinates (x, t) on a uniform lattice, showing that they are difference operators preserving the Lie algebra $so(2, 2)$ as in the continuous case. Therefore some kind of connection between the results of [8] and the quantum $so(2, 2)$ algebra presented here should exist, as was already established for discrete Schrödinger equations and quantum algebras in [9].

Henceforth we follow a parallel procedure with the family $U_\tau(so_{\mu,\nu}(2, 2))$ in two steps. We first introduce a differential-difference realization showing that, indeed, this provides discrete symmetries of a time discretization of the equation (2.5). Secondly, we give a twist map that turns the deformed commutation rules of $U_\tau(so_{\mu,\nu}(2, 2))$ into the Lie commutators of $so_{\mu,\nu}(2, 2)$ but keeping a (deformed) non-cocommutative coproduct in such a manner that a direct relationship with the time discretization of the wave equation studied in [8] from the Lie symmetry approach can finally be established.

A differential-difference realization of $U_\tau(so_{\mu,\nu}(2, 2))$, which under the limit $\tau \rightarrow 0$ gives the continuous conformal realization (2.3), is

$$\begin{aligned} H &= \partial_t & P &= \partial_x \\ K &= -\nu t e^{-\tau \partial_t} \partial_x - \mu x \left(\frac{e^{\tau \partial_t} - 1}{\tau} \right) & D &= -x \partial_x - t \left(\frac{1 - e^{-\tau \partial_t}}{\tau} \right) \\ C_1 &= (\mu x^2 + \nu t^2 e^{-\tau \partial_t}) \left(\frac{e^{\tau \partial_t} - 1}{\tau} \right) + 2\nu x t \partial_x + \tau \nu (x \partial_x + x^2 \partial_x^2) \\ C_2 &= -(\mu x^2 + \nu t^2 e^{-2\tau \partial_t}) \partial_x - 2\mu x t \left(\frac{1 - e^{-\tau \partial_t}}{\tau} \right) + \tau \nu t e^{-2\tau \partial_t} \partial_x. \end{aligned} \quad (4.1)$$

In terms of (4.1) both deformed Casimirs (3.6) vanish. The generators $\{K, H, P\}$ close a deformed subalgebra, the Casimir of which is given by

$$E_\tau = \nu P^2 - \mu \left(\frac{e^{\tau H} - 1}{\tau} \right)^2. \quad (4.2)$$

By introducing the realization (4.1) we find a time discretization of the equation (2.5) on a uniform lattice with x as a continuous variable:

$$E_\tau \Phi(x, t) = 0 \quad \implies \quad \left\{ \nu \frac{\partial^2}{\partial x^2} - \mu \left(\frac{e^{\tau \partial_t} - 1}{\tau} \right)^2 \right\} \Phi(x, t) = 0. \quad (4.3)$$

The generators (4.1) are symmetry operators of (4.3) fulfilling

$$\begin{aligned} [E_\tau, X] &= 0 \quad \text{for } X \in \{K, H, P\} & [E_\tau, D] &= -2E_\tau \\ [E_\tau, C_1] &= 4\nu(t + \tau + \tau x \partial_x) E_\tau & [E_\tau, C_2] &= -4\mu x E_\tau. \end{aligned} \quad (4.4)$$

Hence we conclude that $U_\tau(so_{\mu,\nu}(2, 2))$ is the symmetry algebra of the discrete equation (4.3).

Next, let us consider the so-called minimal twist map, first introduced in [28] for the Jordanian quantum $sl(2, \mathbb{R})$ algebra (here with generators $\{D, H, C_1\}$ and $\nu \neq 0$) and also

used in [9] with other non-standard quantum algebras. This map can be implemented in the whole family $U_\tau(so_{\mu,v}(2, 2))$ as

$$\begin{aligned} \mathcal{H} &= \frac{e^{\tau H} - 1}{\tau} & \mathcal{P} &= P & \mathcal{K} &= K & \mathcal{D} &= D \\ \mathcal{C}_1 &= C_1 - \tau v D^2 & \mathcal{C}_2 &= C_2. \end{aligned} \tag{4.5}$$

These new generators verify the classical commutation rules (2.1), while the coproduct remains deformed as

$$\begin{aligned} \Delta(\mathcal{H}) &= 1 \otimes \mathcal{H} + \mathcal{H} \otimes 1 + \tau \mathcal{H} \otimes \mathcal{H} & \Delta(\mathcal{P}) &= 1 \otimes \mathcal{P} + \mathcal{P} \otimes 1 + \tau \mathcal{P} \otimes \mathcal{H} \\ \Delta(\mathcal{D}) &= 1 \otimes \mathcal{D} + \mathcal{D} \otimes \frac{1}{1 + \tau \mathcal{H}} & \Delta(\mathcal{K}) &= 1 \otimes \mathcal{K} + \mathcal{K} \otimes 1 - \tau v \mathcal{D} \otimes \frac{\mathcal{P}}{1 + \tau \mathcal{H}} \\ \Delta(\mathcal{C}_1) &= 1 \otimes \mathcal{C}_1 + \mathcal{C}_1 \otimes \frac{1}{1 + \tau \mathcal{H}} - 2\tau v \mathcal{D} \otimes \frac{1}{1 + \tau \mathcal{H}} \mathcal{D} + \tau v (\mathcal{D}^2 + \mathcal{D}) \otimes \frac{\tau \mathcal{H}}{(1 + \tau \mathcal{H})^2} \\ \Delta(\mathcal{C}_2) &= 1 \otimes \mathcal{C}_2 + \mathcal{C}_2 \otimes \frac{1}{1 + \tau \mathcal{H}} + 2\tau \mathcal{D} \otimes \frac{1}{1 + \tau \mathcal{H}} \mathcal{K} - \tau^2 v (\mathcal{D}^2 + \mathcal{D}) \otimes \frac{\mathcal{P}}{(1 + \tau \mathcal{H})^2}. \end{aligned} \tag{4.6}$$

The new generator \mathcal{H} verifies $\Delta((1 + \tau \mathcal{H})^a) = (1 + \tau \mathcal{H})^a \otimes (1 + \tau \mathcal{H})^a$ for any real number a , since $(1 + \tau \mathcal{H}) = e^{\tau H}$.

We apply the twist map (4.5) to the realization (4.1) and introduce the time shift operator $T_t = e^{\tau \partial_t}$ and the time difference operator $\Delta_t = (T_t - 1)/\tau$, thus finding

$$\begin{aligned} \mathcal{H} &= \Delta_t & \mathcal{P} &= \partial_x \\ \mathcal{K} &= -vt T_t^{-1} \partial_x - \mu x \Delta_t & \mathcal{D} &= -x \partial_x - t T_t^{-1} \Delta_t \\ \mathcal{C}_1 &= (\mu x^2 + vt^2 T_t^{-2}) \Delta_t + 2vxt T_t^{-1} \partial_x - \tau vt T_t^{-2} \Delta_t \\ \mathcal{C}_2 &= -(\mu x^2 + vt^2 T_t^{-2}) \partial_x - 2\mu xt T_t^{-1} \Delta_t + \tau vt T_t^{-2} \partial_x. \end{aligned} \tag{4.7}$$

In this new basis the Casimir of the subalgebra $\{\mathcal{K}, \mathcal{H}, \mathcal{P}\}$ is the undeformed one (2.4):

$$\mathcal{E} = v\mathcal{P}^2 - \mu\mathcal{H}^2 \tag{4.8}$$

that, written through (4.7), leads again to the discrete equation (4.3):

$$(v\partial_x^2 - \mu\Delta_t^2)\Phi(x, t) = 0. \tag{4.9}$$

The new generators (4.7) are symmetry operators of (4.9) now verifying

$$\begin{aligned} [\mathcal{E}, X] &= 0 \quad \text{for } X \in \{\mathcal{K}, \mathcal{H}, \mathcal{P}\} & [\mathcal{E}, \mathcal{D}] &= -2\mathcal{E} \\ [\mathcal{E}, \mathcal{C}_1] &= 4vt T_t^{-1} \mathcal{E} & [\mathcal{E}, \mathcal{C}_2] &= -4\mu x \mathcal{E}. \end{aligned} \tag{4.10}$$

In this way the relationship between $U_\tau(so_{\mu,v}(2, 2))$ and the symmetries of a time discretization of the wave equation deduced from the Lie theory in [8] clearly arises. In particular, let us denote our generators, contraction parameters and variables by

$$\begin{aligned} \{\mathcal{H}, \mathcal{P}, \mathcal{K}, \mathcal{D}, \mathcal{C}_1, \mathcal{C}_2\} &= \{P_k, P_n, -L, -D, C_k, C_n\} \\ (\mu, v) &= (s^2, +1) & x &= n\sigma & t &= k\tau \end{aligned} \tag{4.11}$$

with $s \neq 0$. If we perform the limits $n \rightarrow \infty$ and $\sigma \rightarrow 0$, subjected to the condition $n\sigma = x$, in the results given in [8] for $m = 0$ (this implies that $\Delta_n \rightarrow \partial_x$, $T_n \rightarrow 1$, $\Delta_k = \Delta_t$, $T_k = T_t$), then we recover the realization (4.7) and the discrete equation (4.9). Consequently, $U_\tau(so_{s^2,1}(2, 2))$ is the quantum symmetry algebra of such an equation and the deformation parameter τ is identified with the time lattice constant in the t coordinate; the space x remains as a continuous variable. Recall that the solutions of (4.9) have also been obtained in [8].

We write down in table 1 the particular equation (4.9) that appears for each quantum algebra in the family $U_\tau(so_{\mu,v}(2, 2))$. It is worth noting that this collective treatment enables a clear view of the contraction limits between these (difference and/or differential) equations, together with their associated symmetry algebras.

Table 2. Hopf subalgebras $\{D, P, C_2\}$ and $\{K, H, P, D\}$, and associated (difference and/or differential) equations of the space-type quantum algebras $U_\sigma(so_{\mu,\nu}(2, 2))$.

$(+, +) U_\sigma(so(2, 2))$	\longrightarrow	$(0, +) U_\sigma(iso(2, 1))$	\longleftarrow	$(-, +) U_\sigma(so(3, 1))$
$U_\sigma(sl(2, \mathbb{R})) U_\sigma(WM)$		$U_\sigma(iso(1, 1)) U_\sigma(WG)$		$U_\sigma(sl(2, \mathbb{R})) U_\sigma(WE)$
$(\Delta_x^2 - \partial_t^2)\Phi = 0$		$\Delta_x^2\Phi = 0$		$(\Delta_x^2 + \partial_t^2)\Phi = 0$
\downarrow		\downarrow		\downarrow
$(+, 0) U_\sigma(iso(2, 1))$	\longrightarrow	$(0, 0) U_\sigma(i'iso(1, 1))$	\longleftarrow	$(-, 0) U_\sigma(iso(2, 1))$
$U_\sigma(sl(2, \mathbb{R})) U_\sigma(WC)$		$U_\sigma(iso(1, 1)) U_\sigma(WA)$		$U_\sigma(sl(2, \mathbb{R})) U_\sigma(WC)$
$\partial_t^2\Phi = 0$		Degenerate equation		$\partial_t^2\Phi = 0$
\uparrow		\uparrow		\uparrow
$(+, -) U_\sigma(so(3, 1))$	\longrightarrow	$(0, -) U_\sigma(iso(2, 1))$	\longleftarrow	$(-, -) U_\sigma(so(2, 2))$
$U_\sigma(sl(2, \mathbb{R})) U_\sigma(WE)$		$U_\sigma(iso(1, 1)) U_\sigma(WG)$		$U_\sigma(sl(2, \mathbb{R})) U_\sigma(WM)$
$(\Delta_x^2 + \partial_t^2)\Phi = 0$		$\Delta_x^2\Phi = 0$		$(\Delta_x^2 - \partial_t^2)\Phi = 0$

5. Space-type quantum algebras and discrete space symmetries

A second natural choice for a non-standard classical r matrix for $so_{\mu,\nu}(2, 2)$, instead of (3.1), is to take

$$r = -\sigma D \wedge P \tag{5.1}$$

where σ is now the deformation parameter. If we follow the same steps described in section 3, we obtain a family of quantum algebras $U_\sigma(so_{\mu,\nu}(2, 2))$ characterized by a primitive generator P (instead of H). The resulting coproduct, commutation rules and universal quantum R matrix are given by

$$\begin{aligned} \Delta(P) &= 1 \otimes P + P \otimes 1 & \Delta(D) &= 1 \otimes D + D \otimes e^{-\sigma P} \\ \Delta(H) &= 1 \otimes H + H \otimes e^{\sigma P} & \Delta(C_2) &= 1 \otimes C_2 + C_2 \otimes e^{-\sigma P} \\ \Delta(K) &= 1 \otimes K + K \otimes 1 - \sigma\mu D \otimes e^{-\sigma P} H \\ \Delta(C_1) &= 1 \otimes C_1 + C_1 \otimes e^{-\sigma P} - 2\sigma D \otimes e^{-\sigma P} K + \sigma^2\mu(D^2 + D) \otimes e^{-2\sigma P} H \end{aligned} \tag{5.2}$$

$$\begin{aligned} [K, H] &= \nu \frac{e^{\sigma P} - 1}{\sigma} & [K, P] &= \mu e^{-\sigma P} H & [H, P] &= 0 & [K, D] &= 0 \\ [D, H] &= H & [D, C_1] &= -C_1 & [H, C_1] &= -2\nu D \\ [D, P] &= \frac{1 - e^{-\sigma P}}{\sigma} & [D, C_2] &= -C_2 - \sigma\mu D^2 & [P, C_2] &= 2\mu D \\ [K, C_1] &= \nu C_2 + \sigma\mu\nu D^2 & [K, C_2] &= \mu C_1 & [P, C_1] &= -e^{-\sigma P} K - K e^{-\sigma P} \\ [H, C_2] &= 2K + \sigma\mu(DH + HD) & [C_1, C_2] &= -\sigma\mu(DC_1 + C_1D) \end{aligned} \tag{5.3}$$

$$\mathcal{R} = \exp\{\sigma P \otimes D\} \exp\{-\sigma D \otimes P\}. \tag{5.4}$$

At the level of Hopf subalgebras of $U_\sigma(so_{\mu,\nu}(2, 2))$, we find that the generators $\{D, P, C_2\}$ give rise to either a quantum $sl(2, \mathbb{R})$ algebra if $\mu \neq 0$ or to a quantum $iso(1, 1)$ algebra if $\mu = 0$. Meanwhile $\{K, H, P, D\}$ close again a quantum Weyl algebra; these Hopf subalgebras are indicated in table 2 for each pair (μ, ν) .

Properties of the family of quantum algebras $U_\sigma(so_{\mu,v}(2, 2))$ are now determined by their primitive generator P , since the product σP implies that the deformation parameter σ has dimensions of *length*; hence we say that these are space-type quantum algebras. Therefore this second quantum deformation leads to a space discretization of the equation (2.5). Explicitly, if we introduce the following differential-difference realization of $U_\sigma(so_{\mu,v}(2, 2))$:

$$\begin{aligned} P &= \partial_x & H &= \partial_t \\ K &= -vt \left(\frac{e^{\sigma\partial_x} - 1}{\sigma} \right) - \mu x e^{-\sigma\partial_x} \partial_t & D &= -x \left(\frac{1 - e^{-\sigma\partial_x}}{\sigma} \right) - t \partial_t \\ C_1 &= (\mu x^2 e^{-2\sigma\partial_x} + vt^2) \partial_t + 2vxt \left(\frac{1 - e^{-\sigma\partial_x}}{\sigma} \right) - \sigma \mu x e^{-2\sigma\partial_x} \partial_t \\ C_2 &= -(\mu x^2 e^{-\sigma\partial_x} + vt^2) \left(\frac{e^{\sigma\partial_x} - 1}{\sigma} \right) - 2\mu xt \partial_t - \sigma \mu (t \partial_t + t^2 \partial_t^2) \end{aligned} \tag{5.5}$$

in the Casimir of the deformed subalgebra $\{K, H, P\}$ given by

$$E_\sigma = v \left(\frac{e^{\sigma P} - 1}{\sigma} \right)^2 - \mu H^2 \tag{5.6}$$

then we obtain a discretization of the equation (2.5) on a uniform space lattice:

$$E_\sigma \Phi(x, t) = 0 \implies \left\{ v \left(\frac{e^{\sigma\partial_x} - 1}{\sigma} \right)^2 - \mu \frac{\partial^2}{\partial t^2} \right\} \Phi(x, t) = 0. \tag{5.7}$$

The quantum algebra $U_\sigma(so_{\mu,v}(2, 2))$ is the symmetry algebra of this equation as the operators (5.5) satisfy

$$\begin{aligned} [E_\sigma, X] &= 0 \quad \text{for } X \in \{K, P, H\} & [E_\sigma, D] &= -2E_\sigma \\ [E_\sigma, C_1] &= 4vt E_\sigma & [E_\sigma, C_2] &= -4\mu(x + \sigma + \sigma t \partial_t) E_\sigma. \end{aligned} \tag{5.8}$$

To unfold the relationship between these discrete space symmetries and the results obtained in [8] from a Lie symmetry approach we consider the twist map for $U_\sigma(so_{\mu,v}(2, 2))$ defined by

$$\begin{aligned} \mathcal{P} &= \frac{e^{\sigma P} - 1}{\sigma} & \mathcal{H} &= H & \mathcal{K} &= K & \mathcal{D} &= D \\ \mathcal{C}_1 &= C_1 & \mathcal{C}_2 &= C_2 + \sigma \mu D^2 \end{aligned} \tag{5.9}$$

that gives rise to the classical commutators (2.1) with the coproduct given by

$$\begin{aligned} \Delta(\mathcal{P}) &= 1 \otimes \mathcal{P} + \mathcal{P} \otimes 1 + \sigma \mathcal{P} \otimes \mathcal{P} & \Delta(\mathcal{H}) &= 1 \otimes \mathcal{H} + \mathcal{H} \otimes 1 + \sigma \mathcal{H} \otimes \mathcal{P} \\ \Delta(\mathcal{D}) &= 1 \otimes \mathcal{D} + \mathcal{D} \otimes \frac{1}{1 + \sigma \mathcal{P}} & \Delta(\mathcal{K}) &= 1 \otimes \mathcal{K} + \mathcal{K} \otimes 1 - \sigma \mu \mathcal{D} \otimes \frac{\mathcal{H}}{1 + \sigma \mathcal{P}} \\ \Delta(\mathcal{C}_1) &= 1 \otimes \mathcal{C}_1 + \mathcal{C}_1 \otimes \frac{1}{1 + \sigma \mathcal{P}} - 2\sigma \mathcal{D} \otimes \frac{1}{1 + \sigma \mathcal{P}} \mathcal{K} + \sigma^2 \mu (\mathcal{D}^2 + \mathcal{D}) \otimes \frac{\mathcal{H}}{(1 + \sigma \mathcal{P})^2} \\ \Delta(\mathcal{C}_2) &= 1 \otimes \mathcal{C}_2 + \mathcal{C}_2 \otimes \frac{1}{1 + \sigma \mathcal{P}} + 2\sigma \mu \mathcal{D} \otimes \frac{1}{1 + \sigma \mathcal{P}} \mathcal{D} - \sigma \mu (\mathcal{D}^2 + \mathcal{D}) \otimes \frac{\sigma \mathcal{P}}{(1 + \sigma \mathcal{P})^2}. \end{aligned} \tag{5.10}$$

Under the map (5.9), the realization (5.5) is transformed into

$$\begin{aligned} \mathcal{P} &= \Delta_x & \mathcal{H} &= \partial_t \\ \mathcal{K} &= -vt \Delta_x - \mu x T_x^{-1} \partial_t & \mathcal{D} &= -x T_x^{-1} \Delta_x - t \partial_t \\ \mathcal{C}_1 &= (\mu x^2 T_x^{-2} + vt^2) \partial_t + 2vxt T_x^{-1} \Delta_x - \sigma \mu x T_x^{-2} \partial_t \\ \mathcal{C}_2 &= -(\mu x^2 T_x^{-2} + vt^2) \Delta_x - 2\mu xt T_x^{-1} \partial_t + \sigma \mu x T_x^{-2} \Delta_x \end{aligned} \tag{5.11}$$

where $T_x = e^{\sigma \partial_x}$ and $\Delta_x = (T_x - 1)/\sigma$. The element E_σ becomes the undeformed \mathcal{E} (4.8), so that the associated differential-difference equation keeps the form of (5.7)

$$(\nu \Delta_x^2 - \mu \partial_t^2) \Phi(x, t) = 0. \quad (5.12)$$

The operators (5.11) are symmetries of this equation since $\{\mathcal{K}, \mathcal{H}, \mathcal{P}\}$ commute with \mathcal{E} and the remaining ones fulfil

$$[\mathcal{E}, \mathcal{D}] = -2\mathcal{E} \quad [\mathcal{E}, \mathcal{C}_1] = 4\nu t \mathcal{E} \quad [\mathcal{E}, \mathcal{C}_2] = -4\mu x T_x^{-1} \mathcal{E}. \quad (5.13)$$

These last results reproduce those found in [8] once we introduce the notation (4.11) and apply the limits $k \rightarrow \infty$, $\tau \rightarrow 0$ (with $k\tau = t$) in the symmetries and equation of [8] (that is, $\Delta_k \rightarrow \partial_t$, $T_k \rightarrow 1$, $\Delta_n = \Delta_x$, $T_n = T_x$). This in turn means that $U_\sigma(so_{s^2,1}(2, 2))$ is the quantum algebra of symmetries of the equation (5.12) on a uniform space lattice with the deformation parameter σ identified with the space lattice constant and t as a continuous variable. The particular equation (5.12) arising for each pair (μ, ν) is written in table 2.

6. 'Duality' and higher dimensions

At a classical level, a remarkable equivalence between the Lie algebras in the family $so_{\mu,\nu}(2, 2)$ (2.1) is provided by the map defined by

$$\begin{array}{lll} H \rightarrow P & P \rightarrow H & K \rightarrow K \\ D \rightarrow D & C_1 \rightarrow -C_2 & C_2 \rightarrow -C_1 \end{array} \quad (6.1)$$

that interchanges the role of the generators $H \leftrightarrow P$ and $C_1 \leftrightarrow C_2$, thus relating the set of graded contractions as

$$so_{\mu,\nu}(2, 2) \leftrightarrow so_{\nu,\mu}(2, 2). \quad (6.2)$$

If the interchange of the two coordinates $x \leftrightarrow t$ is added (so $\partial_x \leftrightarrow \partial_t$), then this algebraic equivalence also works for the vector field realization (2.3) and equation (2.5). This means that, if we consider the classical Lie algebras and associated differential equations arranged as in table 1 by applying the classical limit $\tau \rightarrow 0$ (also as in table 2 for $\sigma \rightarrow 0$), this kind of *duality* corresponds to the reflection in the main diagonal. Thus $so(2, 2)$ and $iiso(1, 1)$ have self-dual structures; meanwhile, for the four Lie algebras $iiso(2, 1)$ this duality interchanges the Weyl subalgebras $\mathcal{WC} \leftrightarrow \mathcal{WG}$ (isomorphic at this dimension) and the differential equations $\partial_t^2 \Phi = 0 \leftrightarrow \partial_x^2 \Phi = 0$ according to the transformation of their contraction parameters $(\pm, 0) \leftrightarrow (0, \pm)$.

When either the time- or space-type quantum deformation is introduced in the family $so_{\mu,\nu}(2, 2)$, it can be checked that the map (6.1) does not lead to a duality as (6.2) for a *single* family of quantum algebras. To implement this duality at a quantum algebra level requires one to consider *both* families $U_\tau(so_{\mu,\nu}(2, 2))$ and $U_\sigma(so_{\mu,\nu}(2, 2))$ simultaneously; then the map (6.1) can be extended by simply interchanging both deformation parameters $\tau \leftrightarrow \sigma$ in such a manner that both families of quantum algebras are related as follows:

$$U_\tau(so_{\mu,\nu}(2, 2)) \leftrightarrow U_\sigma(so_{\nu,\mu}(2, 2)). \quad (6.3)$$

Thus the results presented in table 1 are transformed into those given in table 2, and conversely. For instance, the quantum duality (6.3) interchanges the quantum Weyl subalgebras $U_\tau(\mathcal{WM}) \leftrightarrow U_\sigma(\mathcal{WM})$, $U_\tau(\mathcal{WE}) \leftrightarrow U_\sigma(\mathcal{WE})$, $U_\tau(\mathcal{WG}) \leftrightarrow U_\sigma(\mathcal{WC})$ and $U_\tau(\mathcal{WC}) \leftrightarrow U_\sigma(\mathcal{WG})$, as well as the derivatives $\Delta_t \leftrightarrow \Delta_x$ and $\partial_t \leftrightarrow \partial_x$. Therefore a byproduct of (6.3) is that the expressions (3.11) and (3.13) become a matrix realization and an R matrix for $U_\sigma(so_{\mu,\nu}(2, 2))$ once the map (6.1) has been applied together with the replacements $\mu \leftrightarrow \nu$ and $\tau \leftrightarrow \sigma$.

Consequently, both families of quantum algebras are algebraically equivalent at this (1+1) dimension. In spite of this fact, we consider that the explicit results concerning both families are necessary not only because, from a physical viewpoint they have a different interpretation and allow us to exhibit the duality clearly, but also because they indicate the way to rise to higher dimensions. In this sense, the (1 + 1)D case is somehow exceptional due to the symmetric role that the generators H and P (respectively, the coordinates t and x) play.

We expect that a similar procedure to the one presented in this paper would enable us to construct quantum deformations for the next dimensions (keeping the classical r matrices (3.1) and (5.1) as the seeds of the deformations), particularly for the (3 + 1)D case. The possible quantum $so(4, 2)$ algebras generalizing $U_\tau(so(2, 2))$ and $U_\sigma(so(2, 2))$ would be interpreted as quantum deformations of the conformal algebra of the (3 + 1)D Minkowskian spacetime, giving rise to discretizations of the (3 + 1)D wave equation as

$$\begin{aligned} U_\tau(so(4, 2)) & \quad (\partial_x^2 + \partial_y^2 + \partial_z^2 - \Delta_t^2)\Phi = 0 \\ U_\sigma(so(4, 2)) & \quad (\Delta_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2)\Phi = 0 \end{aligned} \tag{6.4}$$

and fulfilling a sequence of Hopf subalgebra embeddings such as

$$U_\tau(sl(2, \mathbb{R})) \simeq U_\tau(so(2, 1)) \subset U_\tau(so(2, 2)) \subset U_\tau(so(3, 2)) \subset U_\tau(so(4, 2)), \dots \tag{6.5}$$

We have achieved here the first embedding. In this context we remark that in [29] (see also references therein) a systematic construction of a chain of twists applied to the universal envelopings of the semisimple Lie algebras, leading to sequences similar to (6.5), has been introduced. Furthermore the structures (6.4) would be the cornerstone of a scheme of contractions leading to different quantum deformations of the algebras $so(5, 1)$, $so(3, 3)$, $iso(4, 1)$, $iso(3, 2)$, . . . , as well as of their associated differential-difference equations.

To end with we wish to point out that the quantum $iso(2, 1)$ algebras we have obtained can also be interpreted in a kinematical framework as quantum deformations of the (2 + 1)D Poincaré algebra by using a *null-plane basis* [30] with generators $\{P_+, P_1, P_-, E_1, F_1, K_2\}$. If we take, for instance, the Poincaré algebra with contraction parameters $(\mu, \nu) = (0, +1)$, then the relationship between the null-plane generators and the conformal ones is given by

$$\begin{aligned} P_+ &= \frac{1}{\sqrt{2}}P & P_1 &= K & P_- &= -\frac{1}{\sqrt{2}}C_2 \\ E_1 &= -\frac{1}{\sqrt{2}}H & F_1 &= \frac{1}{\sqrt{2}}C_1 & K_2 &= D. \end{aligned} \tag{6.6}$$

This change of basis gives rise to two inequivalent quantum Poincaré algebras: $U_\tau(iso(2, 1)) \supset U_\tau(so(2, 1))$ with E_1 primitive and $U_\sigma(iso(2, 1)) \supset U_\sigma(iso(1, 1))$ with P_+ primitive. These non-standard deformations are different from the so-called null-plane quantum Poincaré algebra [5, 6], the underlying classical r matrix of which is $r = 2z(K_2 \wedge P_+ + E_1 \wedge P_1)$ in the (2 + 1)D case.

Acknowledgment

This work was partially supported by Junta de Castilla y León, Spain (Project CO2/399).

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